Reconstruction of the Potential Function and its Derivatives for the Diffusion Operator

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We solve the inverse nodal problem for the diffusion operator. In particular, we obtain a reconstruction of the potential function and its derivatives using only nodal data. Results are a generalization of Law's and Yang's works.

Key words: Diffusion Operator; Inverse Nodal Problem; Reconstruction Formula.
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1. Introduction

The inverse nodal problem was initiated by McLaughlin [1], who proved that the Sturm-Liouville problem is uniquely determined by any dense subset of the nodal points. Some numerical schemes were given by Hald and McLaughlin [2] for the reconstruction of the potential. Recently Law, Yang and other authors have reconstructed the potential function and its derivatives of the Sturm-Liouville problem from the nodal points [3 – 7].

In this paper, we are concerned with the inverse nodal problem for the diffusion operator on a finite interval. We reconstruct the potential function and all its derivatives by using Law's and Yang's method [7].

The diffusion operator is written as

$$Ly = -y'' + [q(x) + 2\lambda p(x)]y, \tag{1.1}$$

where the function $q(x) \in L^2[0,\pi]$, $p(x) \in L^2[0,\pi]$. Some spectral problems were extensively solved for the diffusion operator in [8 – 11].

Consider the problem

$$L[y] = \lambda^2 y, \tag{1.2}$$

$$y(0) = 1, \quad y'(0) = -h, \tag{1.3}$$

$$y'(\pi,\lambda) + Hy(\pi,\lambda) = 0, \tag{1.4}$$

where h and H are finite numbers.

Let λ_n be the n-th eigenvalue and $0 < x_1^n < \ldots < x_i^n < \pi, i = 1,2,\ldots,n-1$, the nodal points of the n-th eigenfunction. Also let $I_i^n = [x_i^n, x_{i+1}^n]$ be the i-th nodal domain of the n-th eigenfunction and let $l_i^n = |I_i^n| = x_{i+1}^n - x_i^n$ be the associated nodal length. Let $j_n(x)$ be the largest index j such that $0 \le x_j^{(n)} < x$.

Δ denotes the difference operator $\Delta a_i = a_{i+1} - a_i$. Inductively, for $k > 1$, $\Delta^k a_i = \Delta^{k-1} a_{i+1} - \Delta^{k-1} a_i$, and we introduce the difference quotient operator δ:

$$\delta a_i = \frac{a_{i+1} - a_i}{x_{i+1} - x_i} = \frac{\Delta a_i}{l_i} \text{ and } \delta^k a_i = \frac{\delta^{k-1} a_{i+1} - \delta^{k-1} a_i}{l_i}.$$

2. Main Results

Lemma 1. [12] Assume that $q \in L^2[0,\pi]$. Then, as $n \to \infty$ for the problem (1.2) – (1.4),

$$x_i^n = \frac{\left(i - \frac{1}{2}\right)\pi}{\lambda_n} - \frac{h}{2\lambda_n^2} + \frac{1}{2\lambda_n^2}\int_0^{x_i^n} (1 + \cos 2\lambda_n t)[q(t) + 2\lambda_n p(t)]\mathrm{d}t + O\left(\frac{1}{\lambda_n^4}\right), \tag{2.1}$$

$$l_i^n = \frac{\pi}{\lambda_n} + \frac{1}{2\lambda_n^2}\int_{x_i^n}^{x_{i+1}^n} (1 + \cos 2\lambda_n t)[q(t) + 2\lambda_n p(t)]\mathrm{d}t + O\left(\frac{1}{\lambda_n^4}\right). \tag{2.2}$$

Lemma 2. Suppose that $f \in L^2[0,\pi]$. Then for almost every $x \in [0,\pi]$, with $j = j_n(x)$,

$$\lim_{n\to\infty} \frac{\lambda_n}{\pi} \int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t = f(x).$$

Theorem 1. [12] Suppose that $q \in L^2[0,\pi]$, then

$$q(x) = \lim_{n\to\infty} \lambda_n \left(\frac{2\lambda_n^2 l_j^n}{\pi} - 2\lambda_n - 2p(x) \right).$$

Proposition 1. If q is a continuous function, then

(a) $\lim\limits_{n\to\infty} \sqrt{\lambda_n} l_i^{(n)} = \pi$ and $l_i^{(n)} = \frac{1}{n} + O\left(\frac{1}{n^2}\right)$;

(b) $l_{i+k}^{(n)}/l_{i+m}^{(n)} = 1 + O\left(\frac{1}{n}\right)$ for any fixed $k,m \in N$;

(c) $q_m \le \lambda_n - \frac{\pi^2}{(l_i^{(n)})^2} \le q_M$, where $q_m = \min\limits_{[0,\pi]} q(x)$ and $q_M = \max\limits_{[0,\pi]} q(x)$.

Lemma 3. [7] If $q \in C^N[0,\pi]$, then for $k = 1,\dots,N, \Delta^k l_j = O(n^{-(k+3)})$ as $n \to \infty$ and the order estimate is independent of j.

Lemma 4. [7] Let $\Phi_j = \sum\limits_{i=1}^{m} \phi_{j,i}$ with each $\phi_{j,i} = \prod\limits_{p=1}^{k_i} \varphi_{j,i,p}$, where each $\varphi_{j,i,p} \in U_j^{(n)}$. Suppose $\Phi_j = O(n^{-\nu})$ and q is sufficiently smooth. Then $\delta^k \Phi_j = O(n^{-\nu})$ for all $k \in N$.

Lemma 5. [7] Suppose $f \in C^N[0,\pi]$ and $\Phi_j = \int\limits_{x_j}^{x_{j+1}} f(x)\mathrm{d}x$. Then $\delta^k \Phi_j = O(n^{-1})$ for any $k = 0,1,\dots,N$.

Theorem 2. [7] Let $\Phi_m(x_j) = \psi_1(x_j)\psi_2(x_j)\dots\psi_m(x_j)$, where $\psi_i(x_j) = x_{j+k_i}$ and $k_i \in N \cup \{0\}$. If q is C^k on $[0,\pi]$, then

$$\delta^k \Phi_m(x_j) = \left\{ \begin{array}{cc} O(1), & 0 \le k \le m-1, \\ m! + O(n^{-1}), & k = m, \\ O(n^{-2}), & k \ge m+1. \end{array} \right\}$$

Theorem 3. If $q \in C^{N+1}[0,\pi]$, then $q^{(k)}(x) = \delta^k q(x_j) - 2\lambda_n p(x) + O(n^{-1})$ for $k = 0,1,\dots,N$, where $j = j_n(x)$. The order estimate is uniformly valid for compact subsets of $[0,\pi]$.

Remark: For $k = 1,\dots,N$,

$$V_k(x_j) = \begin{bmatrix} 1 & x_j & \cdots & x_j^k \\ 1 & x_{j+1} & \cdots & x_{j+1}^k \\ \vdots & \vdots & & \vdots \\ 1 & x_{j+k} & \cdots & x_{j+k}^k \end{bmatrix}$$

be a $(k+1)\times(k+1)$ Vandermonde matrix. It is well known that

$$\det V_k(x_j) = \prod_{m=1}^{k} \left[\prod_{i=0}^{m-1} \left(\sum_{p=i}^{m-1} l_{j+p} \right) \right].$$

To prove Theorem 3, we need the following lemma.

Lemma 6.

$$\frac{\prod\limits_{m=1}^{k} l_{j+k-m}^m}{\det V_k(x_j)} = \frac{1}{\prod\limits_{m=1}^{k} m!} + O(\frac{1}{n}). \tag{2.3}$$

Next, we consider the following $(k+1)\times(k+1)$ matrix:

$$A = \begin{bmatrix} 1 & x_j & \cdots & x_j^{k-1} & q(x_j) \\ 1 & x_{j+1} & \cdots & x_{j+1}^{k-1} & q(x_{j+1}) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x_{j+k} & \cdots & x_{j+k}^{k-1} & q(x_{j+k}) \end{bmatrix}.$$

After some operations, we obtain

$$\det A = l_j l_{j+1} \dots l_{j+k-1} \cdot \det \begin{bmatrix} 1 & x_j & \cdots & x_j^{k-1} & q(x_j) \\ \delta(1) & \delta x_j & \cdots & \cdot & \delta q(x_j) \\ \vdots & \vdots & & \vdots & \vdots \\ \delta(1) & \delta(x_{j+k-1}) & \cdots & \cdot & \delta q(x_{j+k-1}) \end{bmatrix}$$

$$= (l_j)^k (l_{j+1})^{k-1} \dots l_{j+k-1} \cdot \det \begin{bmatrix} 1 & x_j & \cdots & x_j^{k-1} & q(x_j) \\ \delta(1) & \delta x_j & \cdots & \cdot & \delta q(x_j) \\ \vdots & \vdots & & \vdots & \vdots \\ \delta^k(1) & \delta^k(x_j) & \cdots & \cdot & \delta^k q(x_j) \end{bmatrix}.$$

Let B be the matrix at the right-hand side. By Lemma 4 and Theorem 2,

$$\det B = (\delta x_j)(\delta^2 x_j^2)\dots(\delta^{k-1} x_j^{k-1})(\delta^k q(x_j)) + O\left(\frac{1}{n^2}\right) = \prod_{m=0}^{k-1} (m!)\delta^k q(x_j) + O\left(\frac{1}{n}\right).$$

Proof of Theorem 3: For $k = 1, 2, \ldots, N$, let

$$g(x) = \det \begin{bmatrix} 1 & x & \cdots & x^k & q(x) \\ 1 & x_j & \cdots & x_j^k & q(x_j) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x_{j+k} & \cdots & x_{j+k}^k & q(x_{j+k}) \end{bmatrix}.$$

By Rolle's theorem, $g(x_j) = g(x_{j+1}) = \ldots = g(x_{j+k}) = 0$ implies that there is some $\xi_{1,j+i} \in (x_{j+i}, x_{j+i+1})$ such that $g'(\xi_{1,j+i}) = 0$. When we repeat the process, we can find that $\xi_{k,j} \in (x_j, x_{j+k})$ such that $g^{(k)}(\xi_{k,j}) = 0$. In view of the definition of g, $q^{(k)}(\xi_{k,j}) \det V_k(x_j) = k! \det A$. Hence

$$q^{(k)}(\xi_{k,j}) = (k!)(l_j)^k (l_{j+1})^{k-1} \ldots l_{j+k-1} \frac{\det B}{\det V_k(x_j)}.$$

By Lemma 6 $q^{(k)}(\xi_{k,j}) = \delta^k q(x_j) + O(\frac{1}{n})$, since $q \in C^{k+1}, q^{(k)}(x) = q^{(k)}(\xi_{k,j}) - 2\lambda_n p(x) + O(\frac{1}{n}) = \delta^k q(x_j) - 2\lambda_n p(x) + O(\frac{1}{n})$.

Theorem 4. Suppose that q in (1.1) is C^{N+1} on $[0,\pi]$ $(N \geq 1)$, and let $j = j_n(x)$ for each $x \in [0,\pi]$. Then, as $n \to \infty$,

$$q(x) = \lambda_n \left(\frac{2\lambda_n^2 l_j^n}{\pi} - 2\lambda_n - 2p(x) \right) + O\left(\frac{1}{n}\right),$$

and, for all $k = 1, 2, \ldots, N$,

$$q^{(k)}(x) = \frac{2\lambda_n^{3/2} \delta^k l_j}{\pi} - 2\lambda_n \delta^k p(x_j) - 2\lambda_n p^{(k)}(x) + O(1).$$

Proof: The uniform approximation for q is evident. Suppose that q is continuously differentiable on $[0,\pi]$. Apply the intermediate value theorem on Proposition 1c, then there is some $\xi_j^{(n)} \in (x_j^{(n)}, x_{j+1}^{(n)})$ such that

$$\frac{\lambda_n^2 l_j^{(n)}}{\pi} = \left(1 - \frac{q(\xi_j^{(n)})}{\lambda_n}\right)^{-1/2} = 1 + \frac{q(\xi_j^{(n)})}{2\lambda_n} + O\Big(\frac{1}{\lambda_n^2}\Big).$$

Hence

$$2\lambda_n \left(\frac{\lambda_n^2 l_j^n}{\pi} - 1 \right) - q(\xi_j^{(n)}) = O\left(\frac{1}{n^2}\right),$$

$$2\lambda_n \left(\frac{\lambda_n^2 l_j^n}{\pi} - p(x) \right) + 2\lambda_n p(x) - q(\xi_j^{(n)}) = O\left(\frac{1}{n^2}\right).$$

Applying the mean value theorem, when n is sufficiently large, then

$$q(x) = q(\xi_j^{(n)}) - 2\lambda_n p(x) + O\left(\frac{1}{n}\right).$$

Then we employ a modified Prüfer substitution due to Ashbaugh-Benguria [13] to solve the boundary conditions h and H, $x = 0$ and $x = \pi$, respectively:

$$\begin{cases} y = r(x) \sin \sqrt{\lambda}\, \theta(x), \\ y' = \sqrt{\lambda}\, r(x) \cos \sqrt{\lambda}\, \theta(x), \end{cases}$$

so that

$$\theta' = \cos^2 \sqrt{\lambda}\, \theta(x) - \frac{q(x) \sin^2 \sqrt{\lambda}\, \theta(x)}{\lambda} - 2p(x) \sin^2 \sqrt{\lambda}\, \theta(x). \tag{2.4}$$

Integrating (2.4) from x_j to x_{j+1},

$$\frac{\pi}{\sqrt{\lambda_n}} = \int_{x_j}^{x_{j+1}} \cos^2 \sqrt{\lambda}\, \theta(x) \mathrm{d}x - \frac{1}{\lambda} \int_{x_j}^{x_{j+1}} q(x) \sin^2 \sqrt{\lambda}\, \theta(x) \mathrm{d}x - 2 \int_{x_j}^{x_{j+1}} p(x) \sin^2 \sqrt{\lambda}\, \theta(x) \mathrm{d}x, \tag{2.5}$$

it results by Lemma 5 from (2.5) that

$$\frac{\pi}{\sqrt{\lambda_n}} = -\frac{q(x_j)}{2\lambda_n} l_j + c_j + O\left(\frac{1}{n^{N+4}}\right) - p(x_j) l_j + d_j + O\left(\frac{1}{n^{N+4}}\right) + O\left(\frac{1}{n}\right), \tag{2.6}$$

where

$$c_j = \frac{1}{2\lambda_n} \sum_{k=1}^{N} \frac{q^{(k)}(x_j)}{(k+1)!} l_j^{k+1} = O\left(\frac{1}{n^4}\right),$$

$$d_j = \sum_{k=1}^{N} \frac{p^{(k)}(x_j)}{(k+1)!} l_j^{k+1} = O\left(\frac{1}{n^2}\right).$$

Summarizing from (2.6),

$$q(x_j) = -\frac{2\pi\sqrt{\lambda_n}}{l_j} - 2\lambda_n p(x_j) + 2\lambda_n (c_j + d_j) + O(1).$$

Therefore

$$\delta q(x_j) = -2\pi\sqrt{\lambda_n}\frac{\Delta l_j}{l_j^2 l_{j+1}} - 2\lambda_n \delta p(x_j) + O(1),$$

and so, for $k = 1,2,\ldots,N$,

$$\delta^k q(x_j) = -2\pi\sqrt{\lambda_n}\delta^{k-1}\left(\frac{\Delta l_j}{l_j^2 l_{j+1}}\right) - 2\lambda_n \delta^k p(x_j) + O(1). \quad (2.7)$$

If we use the results of Theorem 3 and Theorem 5, we get

$$q^{(k)}(x) = \delta^k q(x_j) - 2\lambda_n p(x) + O(n^{-1}),$$

$$q^{(k)}(x) = \frac{2\lambda_n^{3/2}\delta^k l_j}{\pi} - 2\lambda_n \delta^k p(x_j) - 2\lambda_n p(x) + O(1).$$

Theorem 5. Assume that q is C^{N+1} on $[0,\pi]$. Then, for $k = 1,2,\ldots,N$,

$$\delta^k q(x_j) = \frac{2\lambda_n^{3/2}\delta^k l_j}{\pi} - 2\lambda_n \delta^k p(x_j) + O(1).$$

The estimate is independent of j.

Proof: In view of derivations and the fact that $\delta l_j = \frac{\Delta l_j}{l_j} = \frac{O(n^{-4})}{O(n^{-1})} = O(n^{-3})$, it suffices to show that

$$\delta^{k-1}\left(\frac{\pi^2\Delta l_j}{l_j^2 l_{j+1}}\right) = -\delta^{k-1}\left(\frac{\lambda_n \Delta l_j}{l_j}\right) + O\left(\frac{1}{n^3}\right). \quad (2.8)$$

But

$$\delta\left(\frac{\lambda_n\Delta l_j}{l_j} - \frac{\pi^2\Delta l_j}{l_j^2 l_{j+1}}\right) = \delta\left[\left(\lambda_n - \frac{\pi^2}{l_j l_{j+1}}\right)\frac{\Delta l_j}{l_j}\right]$$
$$= \pi^2\frac{(\Delta l_j + \Delta l_{j+1})}{l_j^2 l_{j+1} l_{j+2}} + \left(\lambda_n - \frac{\pi^2}{l_j l_{j+1}}\right)\delta l_j$$
$$= O\left(\frac{1}{n^3}\right).$$

Thus, (2.8) follows by Lemma 4. If we write (2.8) in (2.7), then

$$\delta^k q(x_j) = -2\pi\sqrt{\lambda_n}\left[-\delta^{k-1}\Big(\frac{\lambda_n\Delta l_j}{\pi^2 l_j}\Big) + O\Big(\frac{1}{n^3}\Big)\right] - 2\lambda_n\delta^k p(x_j) + O(1), \quad (2.9)$$

$$\delta^k q(x_j) = \frac{2\lambda_n^{3/2}\delta^k l_j}{\pi} - 2\lambda_n \delta^k p(x_j) + O(1).$$

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